Thermal phase slips in superconducting films

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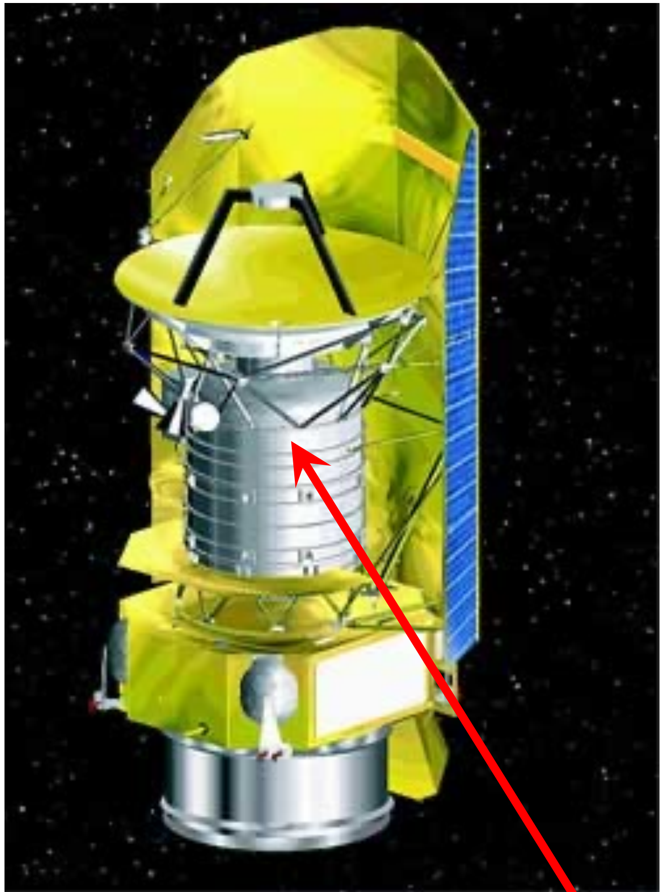
- Superconducting photon detectors
- Thermal phase slips
    - 0D: Josephson junction
    - 1D: Langer-Ambegaokar theory
    - 2D: Status quo
- Analytical solution in 2D at $I \to I_c$
    - Stream function
    - Boussinesq equation
    - Its instanton solution
- Away from I_c & topological transition between instantons

Superconducting Nanowire Single-Photon Detectors

Herschel Space Observatory
(2009—2013)



Operates at 1.4 K

Thought to be
star formation in the early Universe

far infrared and submillimeter
wavebands (55–672 μm)

dirty superconductor inside

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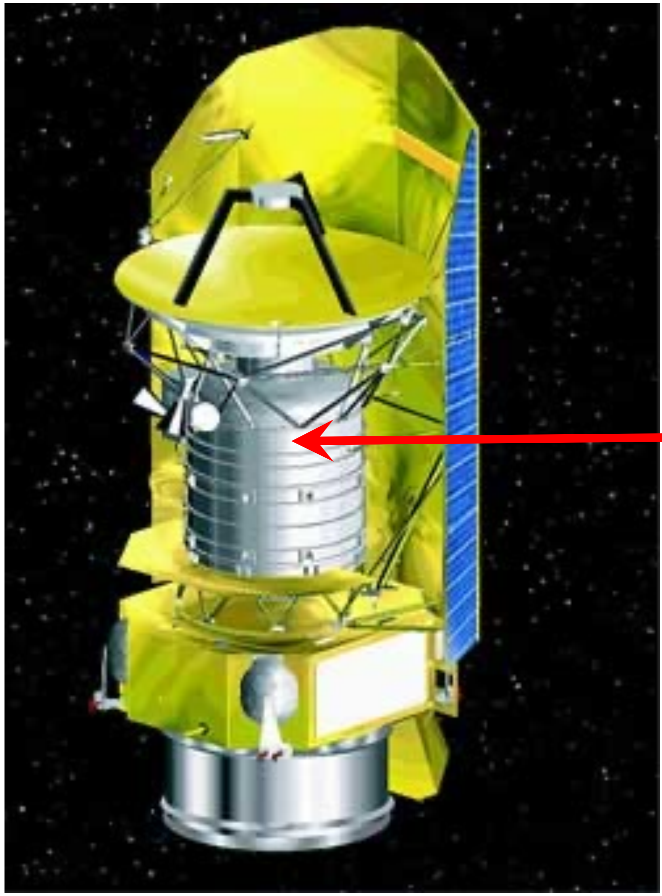

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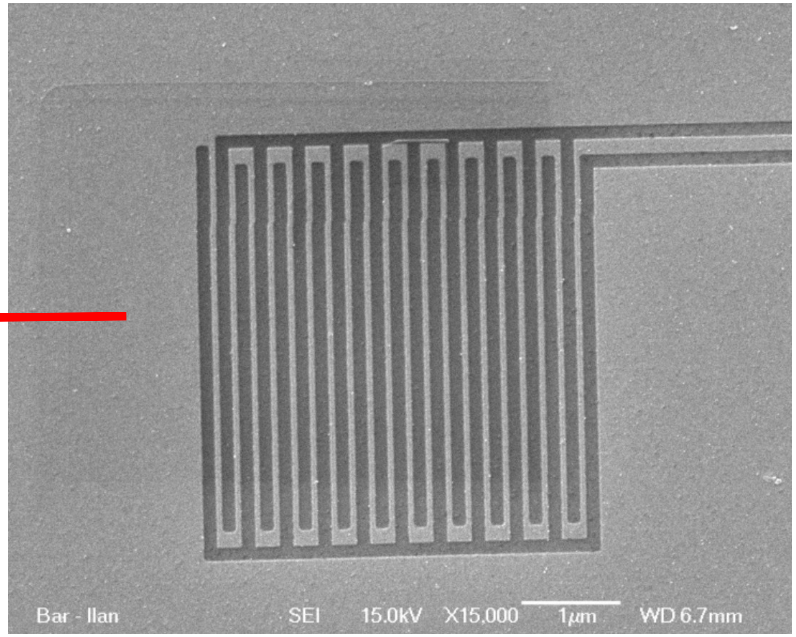
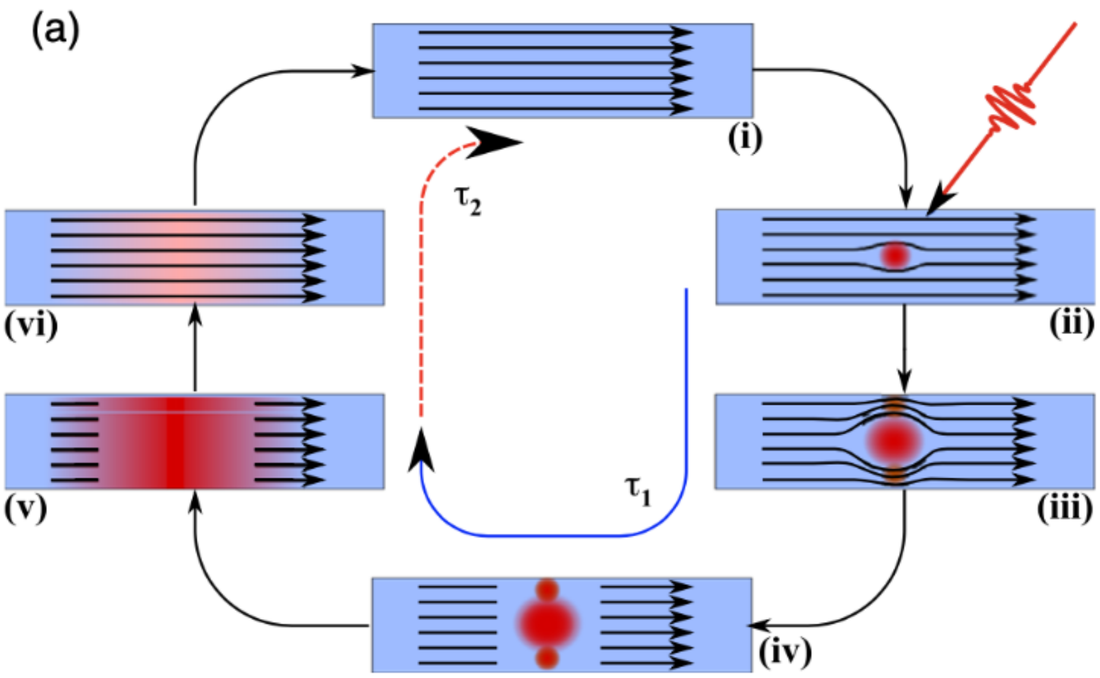
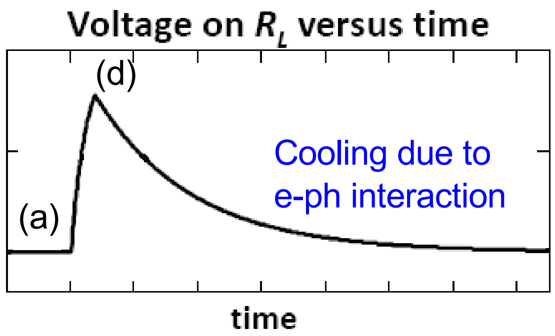


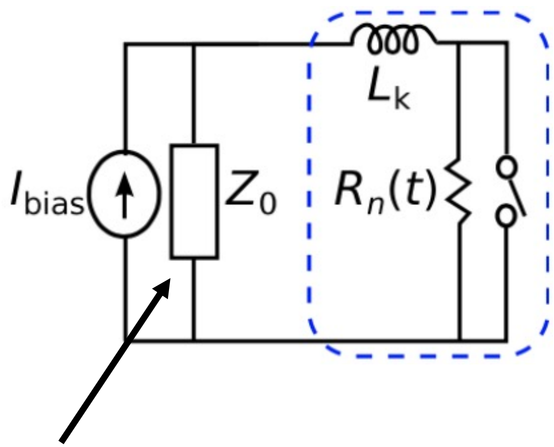
meandered NbN wire

Superconducting Nanowire Single-Photon Detectors

Idea: A. D. Semenov, G. N. Goltsman, A. A. Korneev (2001)

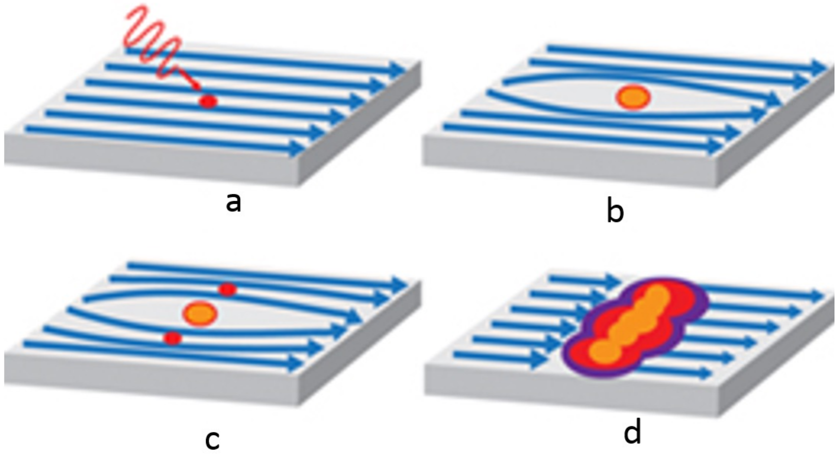


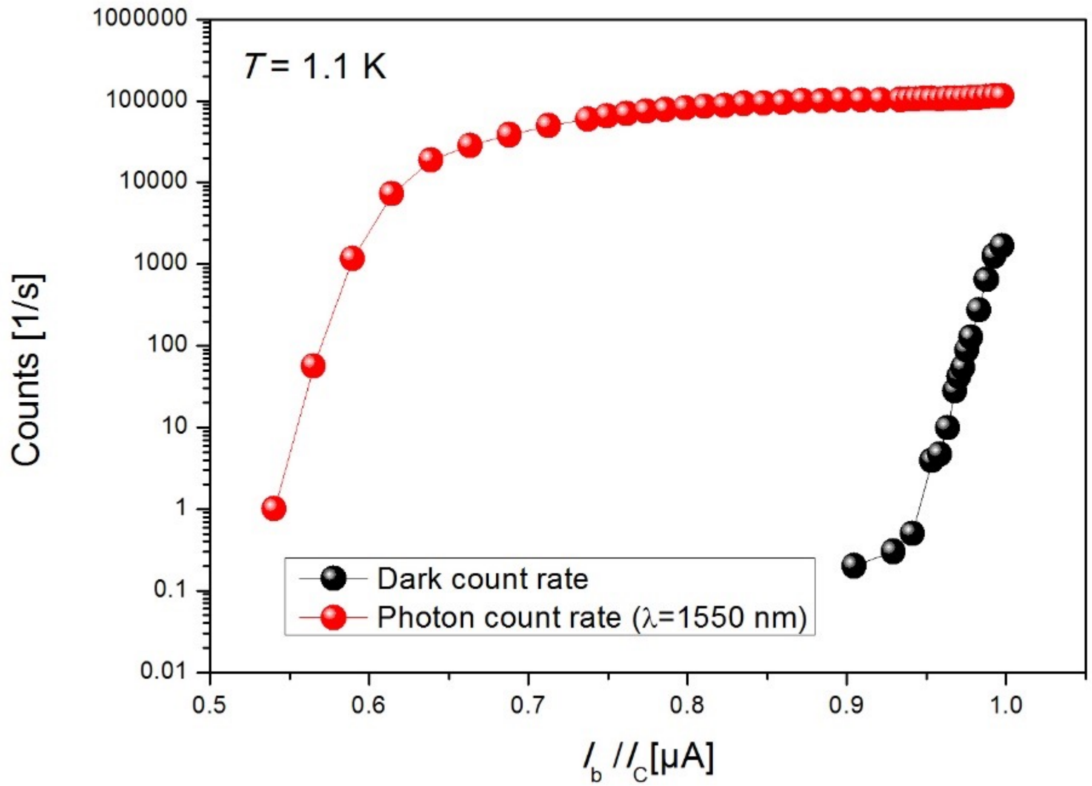




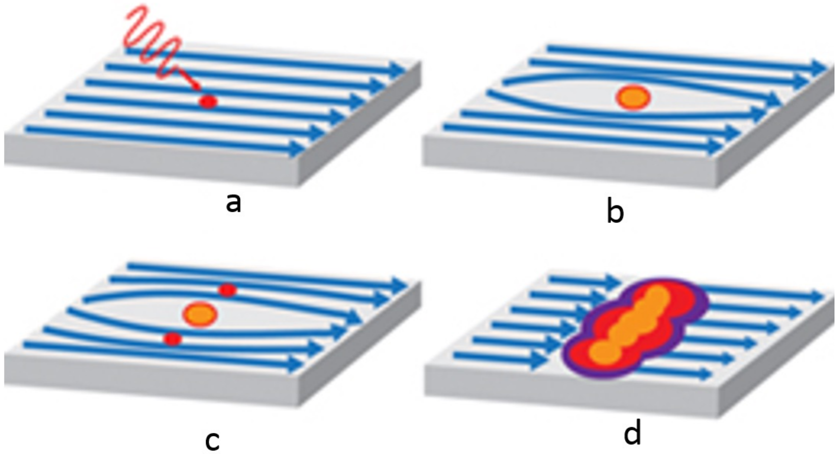
shunt
resistance

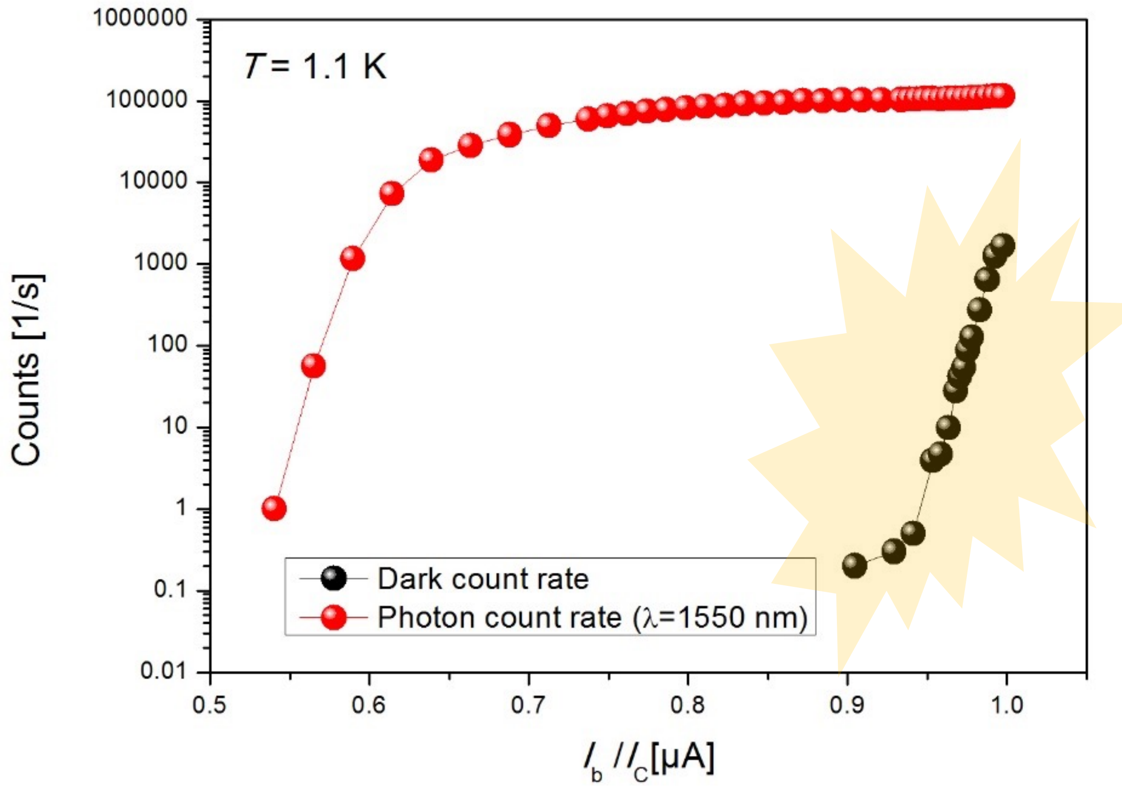
Photon counts & dark counts in SNSPDs





Photon counts & dark counts in SNSPDs





dark counts
=
thermal phase slips

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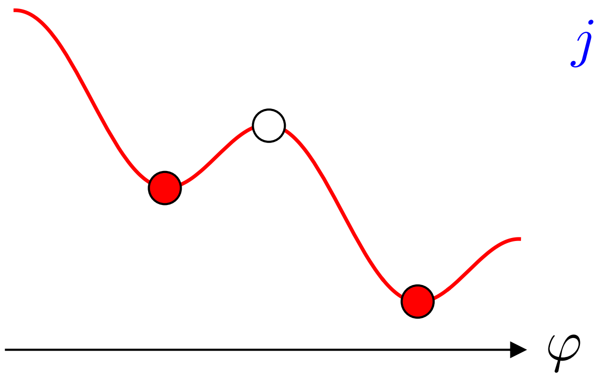
Thermal phase slips in a Josephson junction (0D)

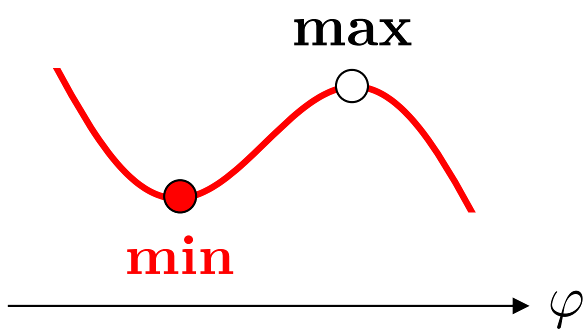
Current-biased JJ

$$U(\varphi) = E_J(1 - \cos\varphi) - E_J j \varphi$$

$$j = I/I_c$$



Activation rate: $e^{-\Delta F/T}$



$$\Delta F = U(\varphi_{\max}) - U(\varphi_{\min})$$
$$= 2E_J\left[\sqrt{1 - j^2} - j \arccos j\right]$$

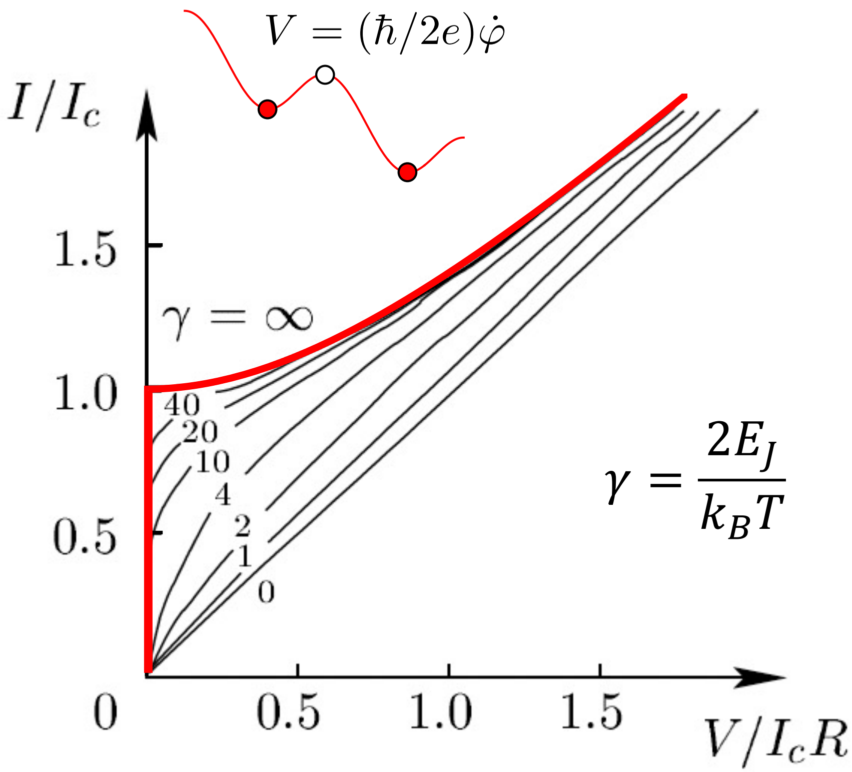
In the limit $I \to I_c$, the barrier has a power-law dependence:

$$\Delta F^{0D} \approx \frac{2^{5/2}}{3} E_J (1 - I/I_c)^{3/2}$$

Thermal phase slips: what's then?

- **Zero-frequency measurements**

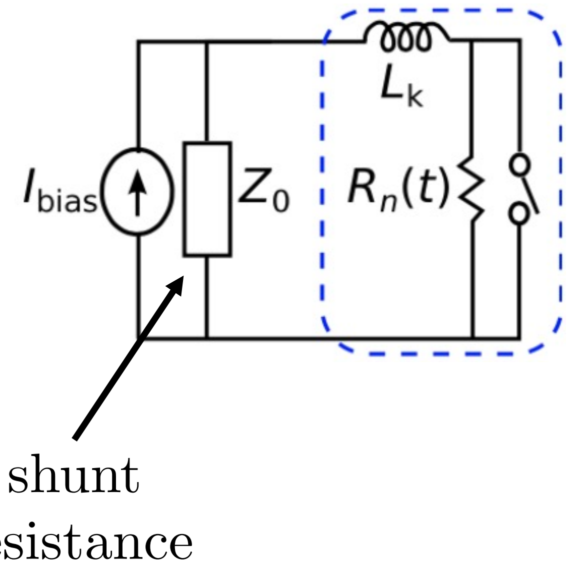
finite resistance



Wires: resistive state............

- **Time-resolved measurements**

requires a mechanism to recover the initial superconducting state implemented in SNSPDs



resolving **individual** dark counts

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Supercurrent state in wires

Ginzburg-Landau free energy:

$$F = C \int dx \left(|\nabla\Delta|^2 - |\Delta|^2 + |\Delta|^4/2\right)$$

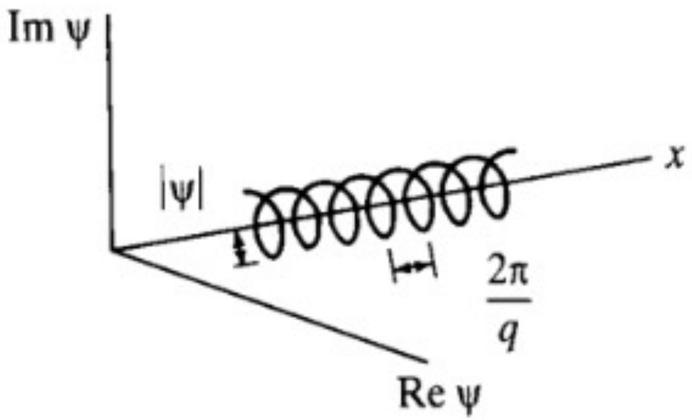
Ginzburg-Landau equation for $\Delta(\mathbf{r}) = |\Delta| e^{i\varphi}$:

$$-\nabla^2\Delta - \Delta + |\Delta|^2\Delta = 0$$

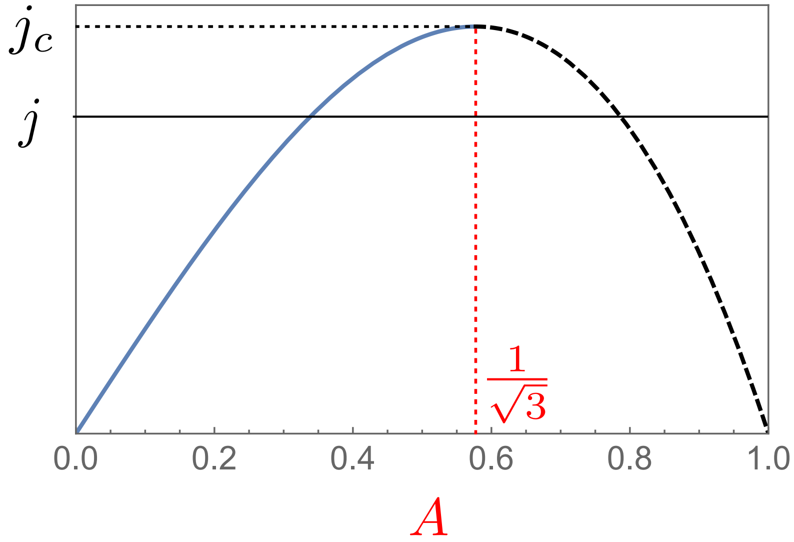
Supercurrent:

$$\mathbf{j} = \eta|\Delta|^2\nabla\varphi$$

Supercurrent solution: $\Delta(x) = \sqrt{1 - A^2}\, e^{iAx}$



$$j = (1 - A^2)A$$



Phase slips: What is the barrier?

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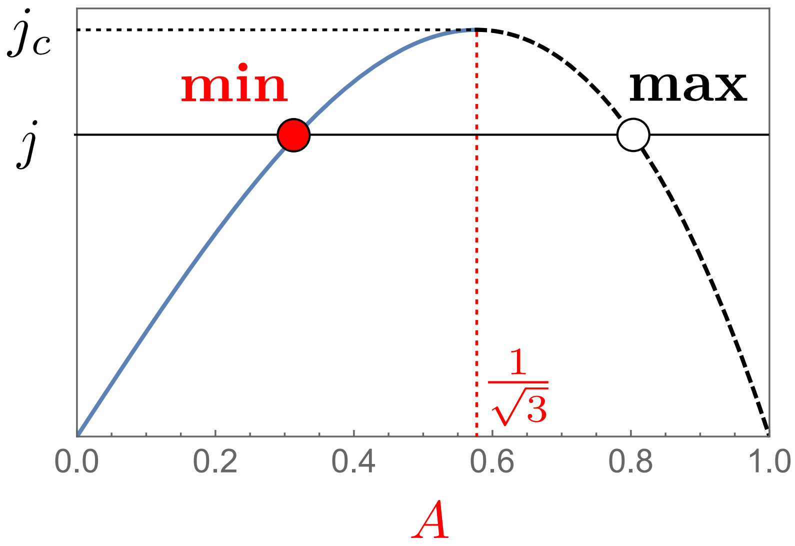

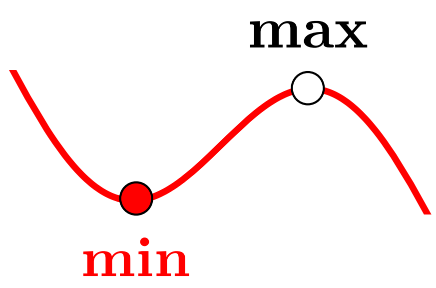

Naive barrier is extensive:

$$\Delta F \propto \int_0^L dx = L$$

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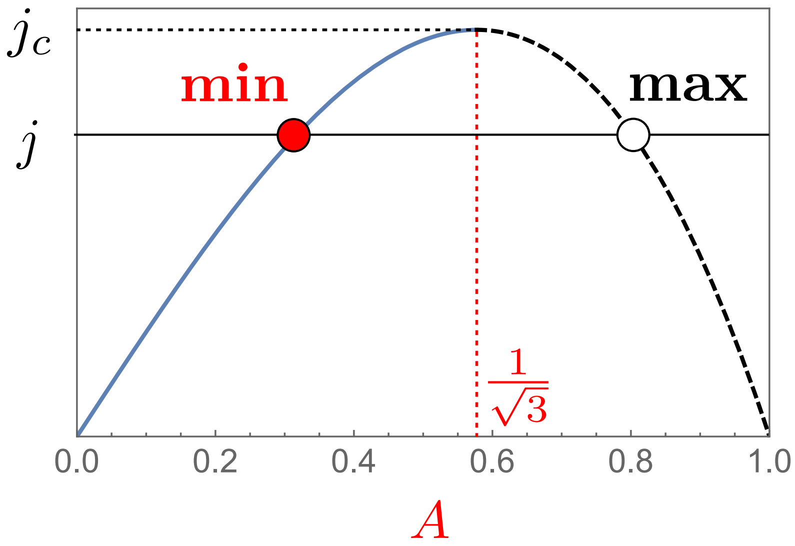

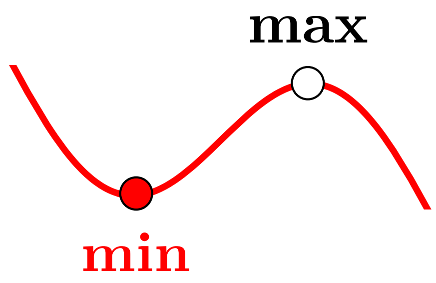

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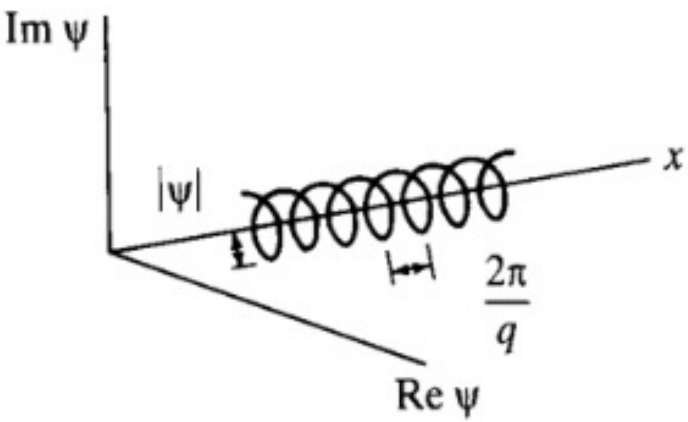
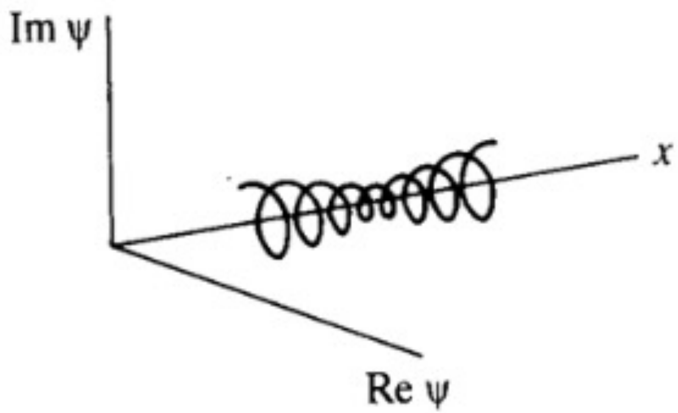
The way out: a non-uniform solution [Langer and Ambegaokar (1967)]





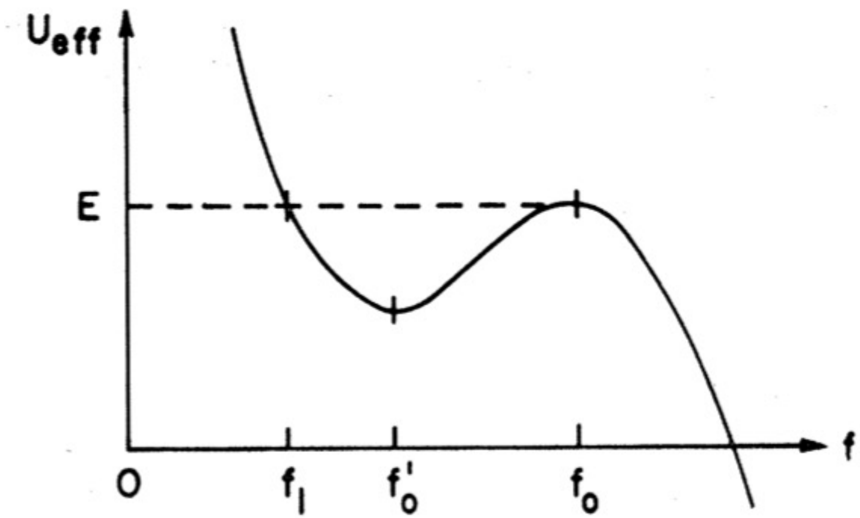
Langer-Ambegaokar instanton

Ginzburg-Landau equation (here $f = |\Delta|$):

$$-f'' - (1 - \varphi'^2)f + f^3 = 0, \qquad (f^2\varphi')' = 0$$

Current conservation $j = f^2\varphi' = \text{const}$ allows to exlude the phase and to arrive at a mechanical problem

$$f'' = -\frac{dU_{\text{eff}}}{df}, \qquad U_{\text{eff}}(f) = \frac{f^2}{2} + \frac{J^2}{2f^2} - \frac{f^4}{4}$$



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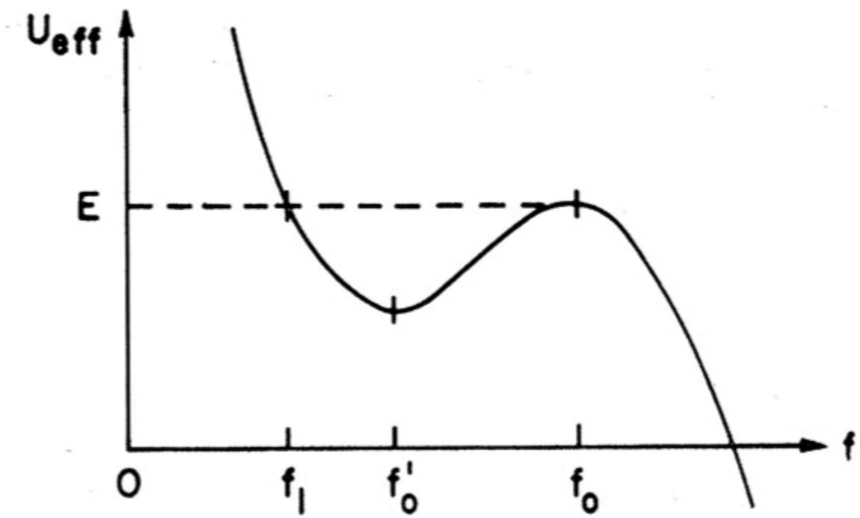
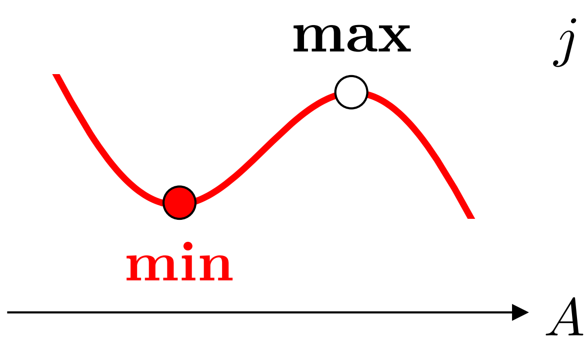


In the uniform case:



$j = f^2 A$

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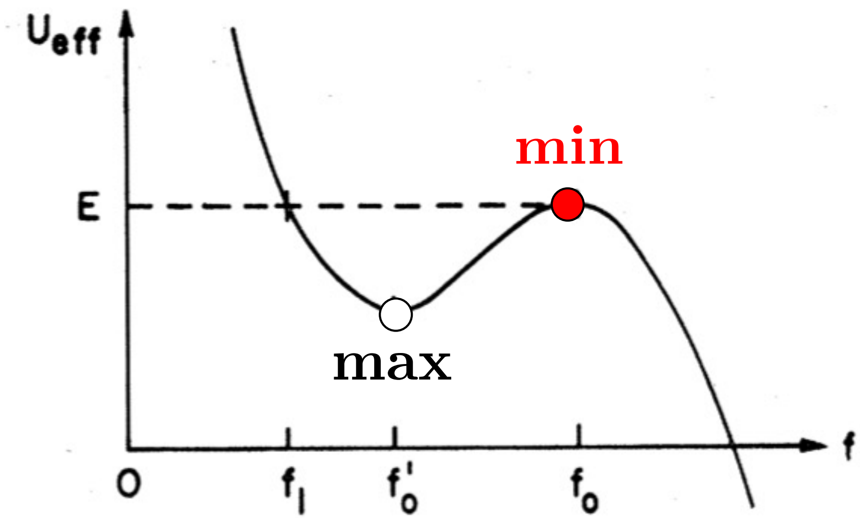

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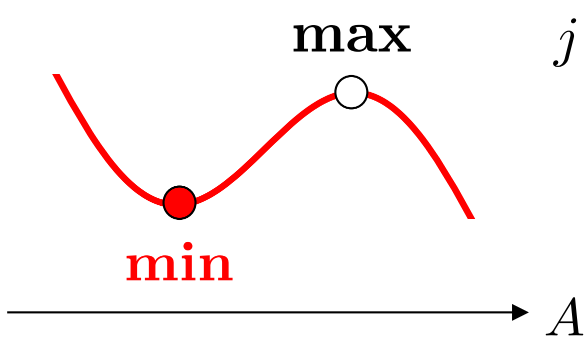

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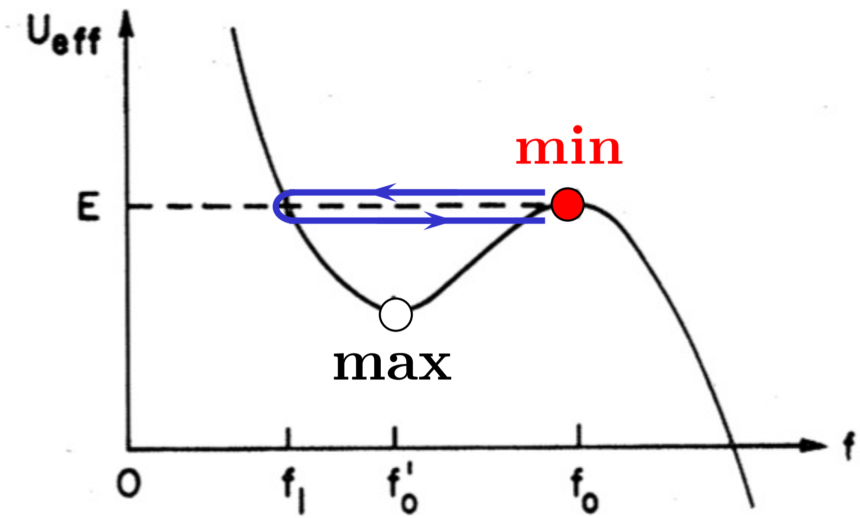

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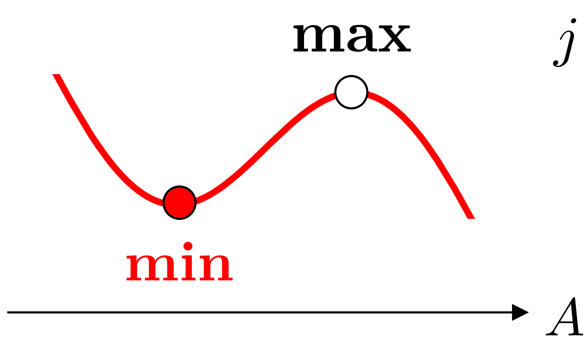

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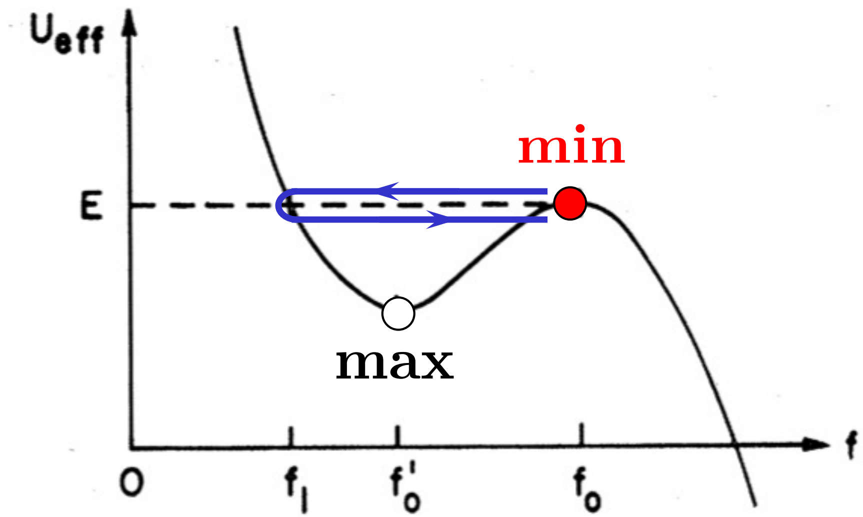

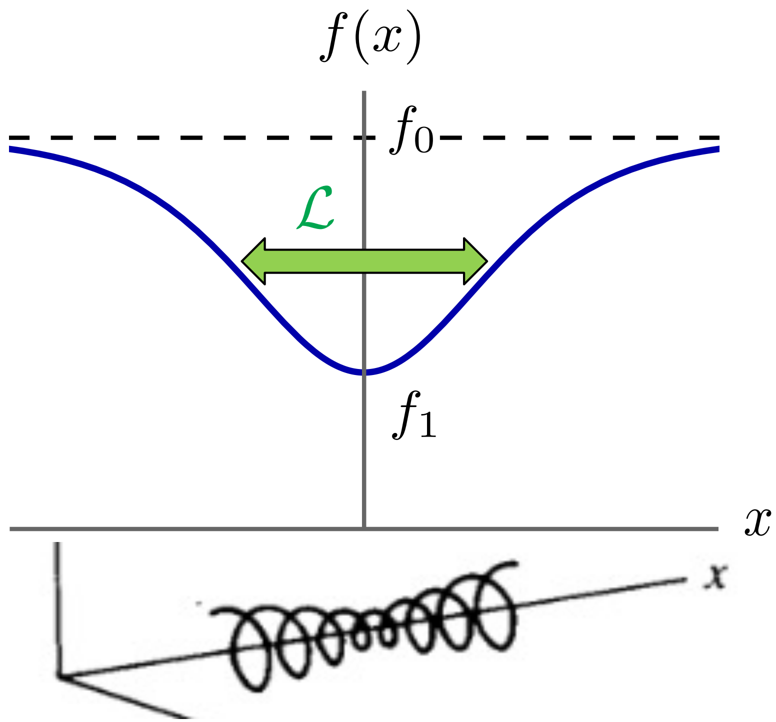

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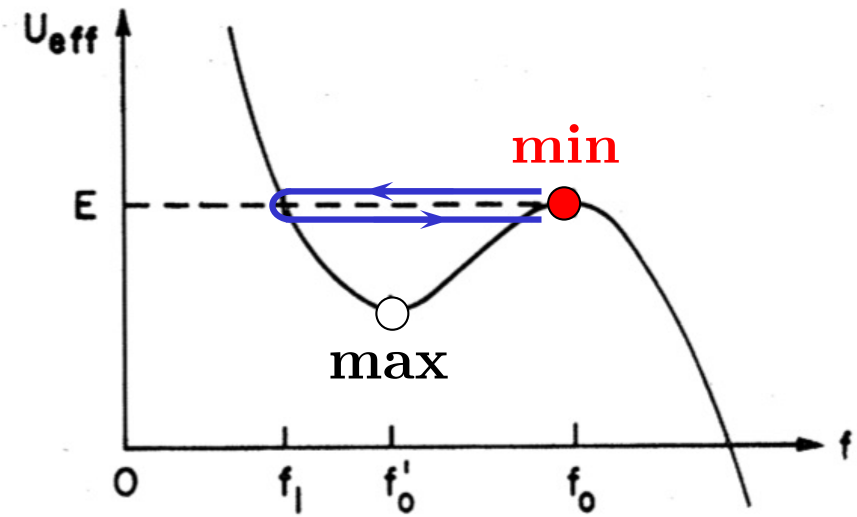

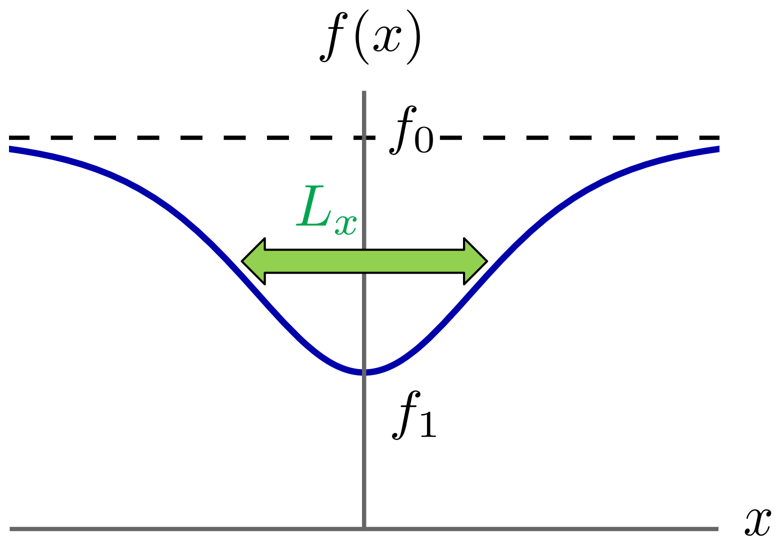

Analytic solution:

$$f(x) = f_0 - (f_0 - f_1)/\cosh^2(x/L_x)$$

Instanton size:

$$L_x \sim \xi(T)(1 - I/I_c)^{-1/4}$$

Energy barrier:

$$\Delta F^{1D} \propto (1 - I/I_c)^{3/2} L_x$$

$$\propto (1 - I/I_c)^{5/4}$$

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GL instanton in 2D

Ginzburg-Landau equation for the complex $\Delta(\mathbf{r})$:

$$-\nabla^2\Delta - \Delta + |\Delta|^2\Delta = 0$$

Boundary condition at infinity: $\mathbf{j} = |\Delta|^2\nabla\varphi \to j_0\mathbf{e}_x$

Current conservation: $\operatorname{div}\mathbf{j} = \nabla(|\Delta|^2\nabla\varphi) = 0$

- 1D: $|\Delta|^2\varphi' = j_0 = \text{const}$

  closed equation for $|\Delta(x)|$
  [Langer, Ambegaokar (1964)]

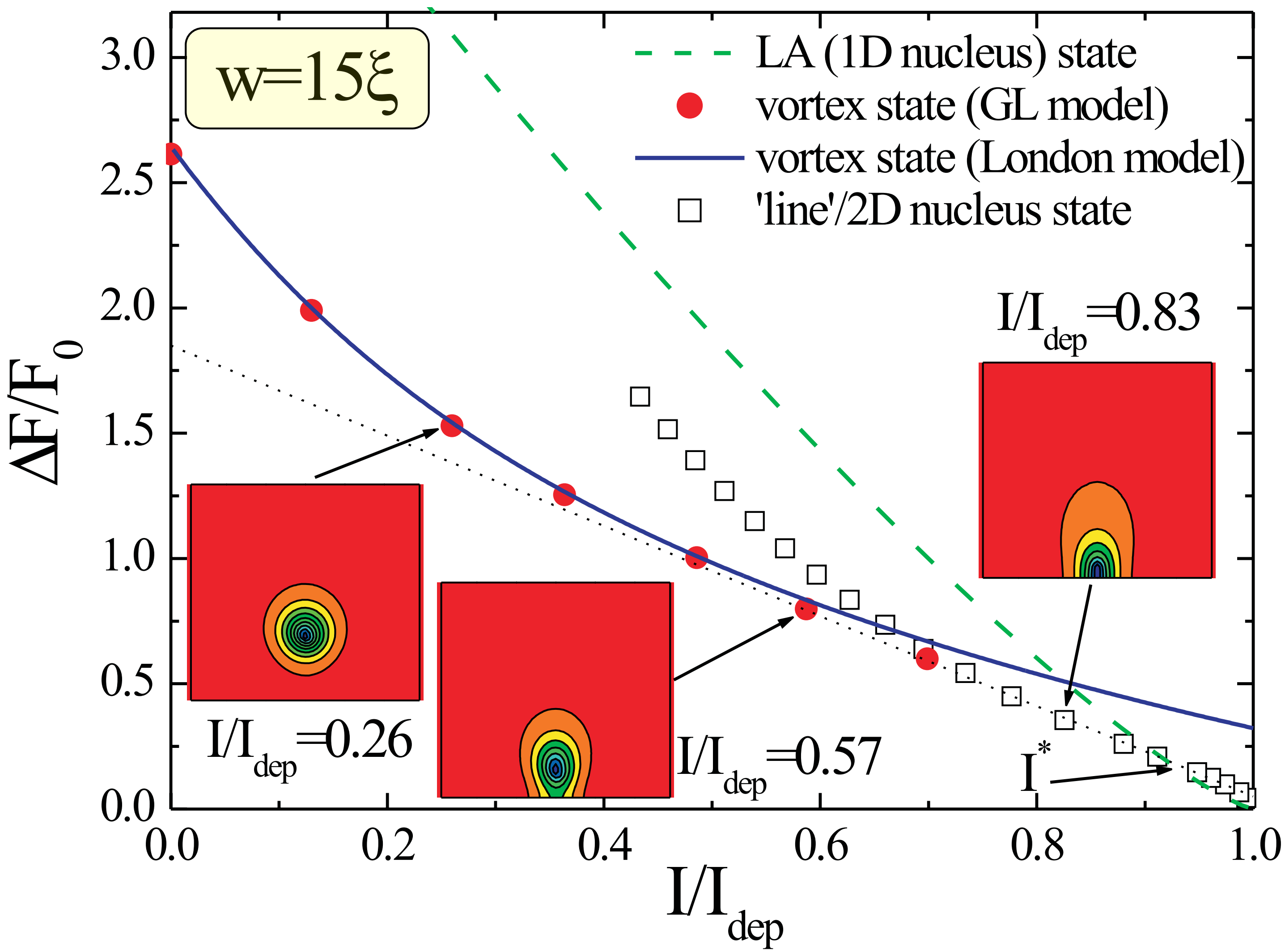
- 2D: **nonlocality**

  coupled PDEs
  for $|\Delta(x,y)|$ and $\varphi(x,y)$

  No way?

GL instanton in 2D strips: numerics

L. N. Bulaevskii, M. J. Graf, C. D. Batista, V. G. Kogan (2011)
D. Y. Vodolazov (2012)



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Stream function

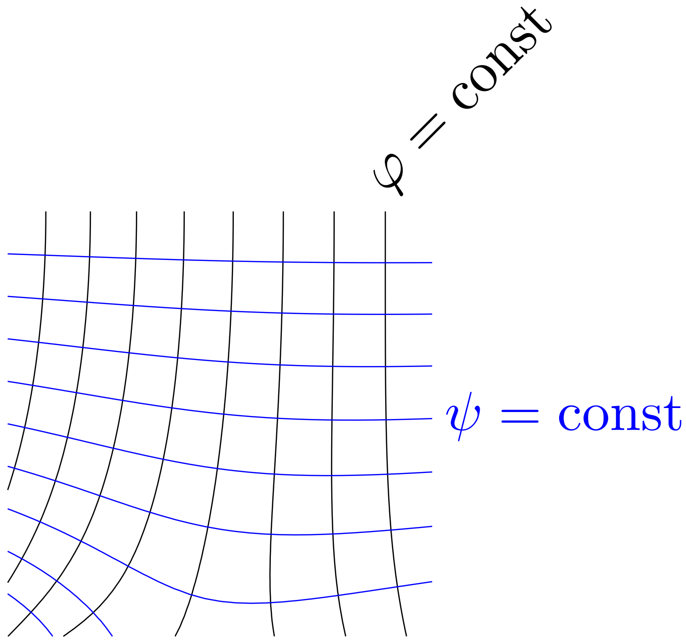
Description of an incompressible flow by the stream function $\psi(\mathbf{r})$

$$\mathbf{j} = (\psi_y, -\psi_x)$$

- Automatically resolves $\operatorname{div} \mathbf{j} = 0$
- How to get $|\Delta(\mathbf{r})|$ and $\varphi(\mathbf{r})$?

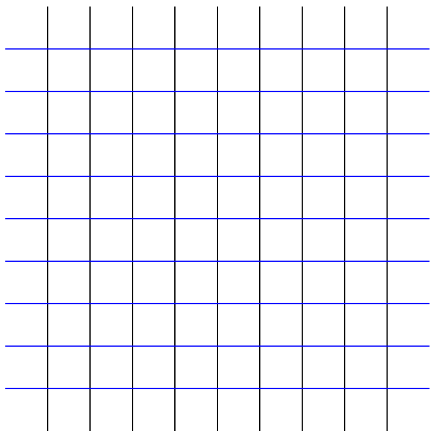
$$\boxed{\nabla\psi\nabla\varphi = 0}$$

That is, ψ and φ are orthogonal curvilinear coordinates on the plane



Uniform current:

$\psi = j_0 y$ and $\varphi = Ax$

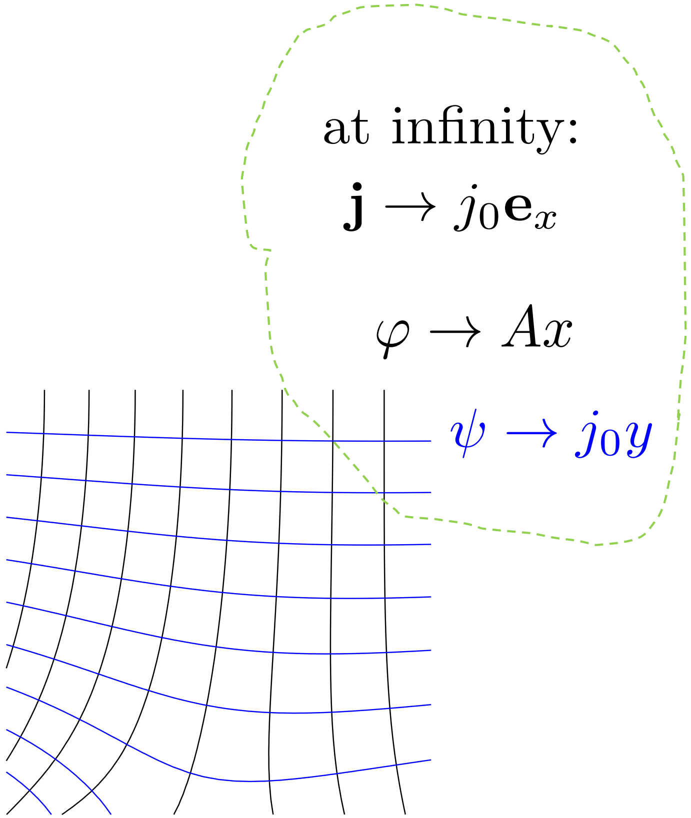


From stream function to order parameter

1. Determine the **phase** φ

   by solving $\nabla\psi\nabla\varphi = 0$

   (nonlocal)



Solution for $\phi(\mathbf{r})$
exists and is unique
provided $\nabla\psi \neq 0$
(no vortices)

2. Determine the **modulus** $|\Delta|$ from $\mathbf{j} = |\Delta|^2\nabla\varphi$ (local)

Result: nonlocal (numerical) and nonlinear $F[\psi]$

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Near the critical current

At $I \to I_c$, instanton is a weak perturation of the uniform solution

- Stream function: $\psi = j_0(y + f_y)$, and f is small
- Phase: $\varphi = A(x + g)$ $\quad g = g^{(1)} + g^{(2)} + \ldots$, where $g^{(n)} \propto f^n$
- Modulus: $|\Delta|^2 = \Delta_0^2[1 + (1 + f_{yy})/(1 + g_x)]$

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$$\nabla\psi\nabla\varphi = 0 \quad \Longrightarrow \quad f_{xy}(1 + g_x) + (1 + f_{yy})g_y = 0$$

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$$\nabla\psi\nabla\varphi = 0 \quad \Longrightarrow \quad f_{xy}(1 + g_x) + (1 + f_{yy})g_y = 0$$

- 1 order: $g_y^{(1)} = -f_{xy}$ $\longrightarrow$ $g^{(1)} = -f_x$ (local!)
- 2 order: $g_y^{(2)} = (f_{xx} + f_{yy})f_{xy}$
- 3 order: $g_y^{(3)} = -(f_{xx} + f_{yy})f_{xy}f_{yy} - g_x^{(2)} f_{xy}$

(2 and 3 order: nonlocal)

Near the critical current

$$\varepsilon = 1 - 3A^2 \approx \sqrt{(8/3)(1 - I/I_c)} \ll 1$$

- Quadratic terms: local $\varepsilon f_{xx}^2,\ f_{xy}^2,\ f_{yy}^2,\ f_{xxx}^2,\ f_{xxy}^2,\ f_{xyy}^2,\ f_{yyy}^2$
- Cubic terms: plenty of local and nonlocal terms

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Length scales: $L_x \sim \frac{1}{\sqrt{\varepsilon}}$ $\quad L_y \sim \frac{1}{\varepsilon}$ $\quad L_y \gg L_x$

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$$F = C \int dx\, dy \left(\frac{\varepsilon}{3} f_{xx}^2 + \frac{2}{9} f_{xy}^2 + \frac{1}{6} f_{xxx}^2 + \frac{2}{9} f_{xx}^3 \right)$$

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- Quadratic terms: local $\varepsilon f_{xx}^2,\ f_{xy}^2,\ f_{yy}^2,\ f_{xxx}^2,\ f_{xxy}^2,\ f_{xyy}^2,\ f_{yyy}^2$
- Cubic terms: plenty of local and nonlocal terms

Length scales: $L_x \sim \frac{1}{\sqrt{\varepsilon}}$ $L_y \sim \frac{1}{\varepsilon}$ $L_y \gg L_x$

$$F = C \int dx\, dy \left(\frac{\varepsilon}{3} f_{xx}^2 + \frac{2}{9} f_{xy}^2 + \frac{1}{6} f_{xxx}^2 + \frac{2}{9} f_{xx}^3 \right)$$

Rescaling $F = 2^{1/2} 3^{-3/2} \varepsilon^{3/2} S C$

$$S = \int d\bar{x}\, d\bar{y} \left(\frac{1}{2} \bar{f}_{\bar{x}\bar{x}}^2 + \frac{1}{2} \bar{f}_{\bar{x}\bar{y}}^2 + \frac{1}{2} \bar{f}_{\bar{x}\bar{x}\bar{x}}^2 + \frac{1}{3} \bar{f}_{\bar{x}\bar{x}}^3 \right)$$

$f = \bar{f}/2$

Boussinesq equation

$$S = \int d\bar{x}\, d\bar{y} \left(\frac{1}{2} \bar{f}^2_{\bar{x}\bar{x}} + \frac{1}{2} \bar{f}^2_{\bar{x}\bar{y}} + \frac{1}{2} \bar{f}^2_{\bar{x}\bar{x}\bar{x}} + \frac{1}{3} \bar{f}^3_{\bar{x}\bar{x}} \right)$$

Saddle-point equation written in terms of $u = \bar{f}_{\bar{x}\bar{x}}$:

$$u_{\bar{x}\bar{x}} + u_{\bar{y}\bar{y}} - u_{\bar{x}\bar{x}\bar{x}\bar{x}} + (u^2)_{\bar{x}\bar{x}} = 0$$

Boussinesq equation in the elliptic form

In its original (hyperbolic) formulation, with $\bar{y} = it$ and t being time, this equation describes propagation of shallow water waves in the long-wavelength limit [J. Boussinesq (1872)].

- Nonlinear PDE **integrable** by the inverse scattering method [V. E. Zakharov, S. V. Manakov, S. P. Novicov, and L. P. Pitaevsky, *Theory of Solitons: The Inverse Scattering Method* (1984)]
- Existence of solitary waves (**solitons**)

- Superconducting photon detectors
- Thermal phase slips
  - 0D: Josephson junction
  - 1D: Langer-Ambegaokar theory
  - 2D: Status quo
- Analytical solution in 2D at $I \to I_c$
  - Stream function
  - Boussinesq equation
  - Its instanton solution
- Away from I_c & topological transition between instantons

Instanton in the Boussinesq equation

$$u_{\bar{x}\bar{x}} + u_{\bar{y}\bar{y}} - u_{\bar{x}\bar{x}\bar{x}\bar{x}} + (u^2)_{\bar{x}\bar{x}} = 0$$

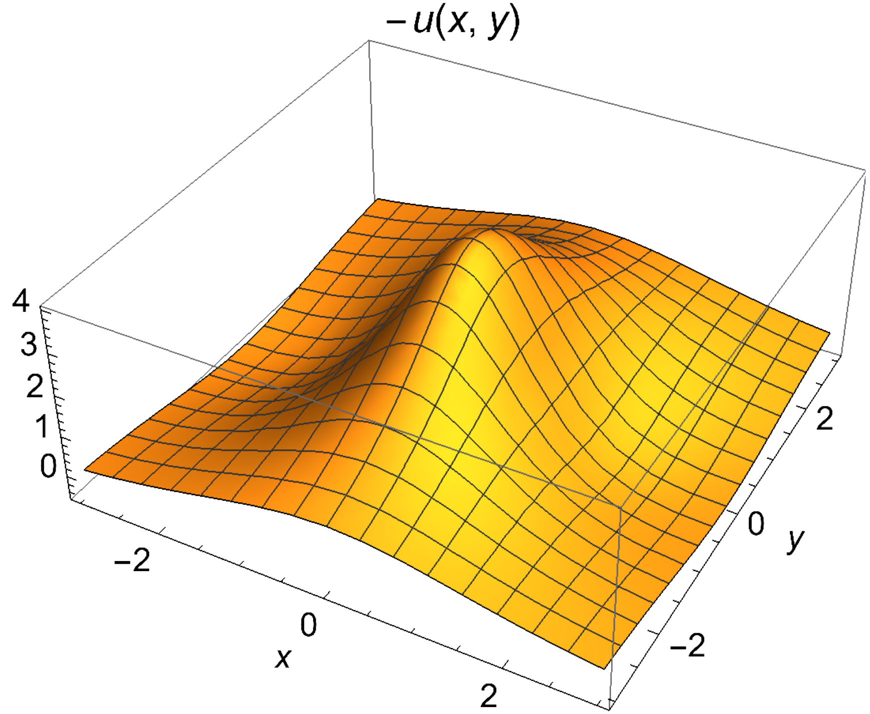
We need a nontrivial localized solution vanishing at $r \to \infty$

It can be found in the form inspired by Hirota:

$$u_{\mathrm{B}} = -6\partial_{\bar{x}}^2 \ln(\bar{x}^2 + \bar{y}^2 + 3)$$

$$\bar{f}_{\mathrm{B}} = -6 \ln(\bar{x}^2 + \bar{y}^2 + 3)$$

Instanton action
$S = 8\pi$



Free energy barrier

$$\boxed{\Delta F^{\mathrm{2D}} = c_2 \varepsilon_{\mathrm{cond}} d \xi^2 (1 - I/I_c)^{3/4}}$$

$$c_2 = 2^{27/4} 3^{-9/4} \pi = 28.55$$

Free energy barrier

$$\boxed{\Delta F^{2\mathrm{D}} = c_2 \varepsilon_{\mathrm{cond}} d\xi^2 (1 - I/I_c)^{3/4}}$$

$$c_2 = 2^{27/4} 3^{-9/4} \pi = 28.55$$

$$\Delta F^{1\mathrm{D}} = c_1 \varepsilon_{\mathrm{cond}} dw\xi (1 - I/I_c)^{5/4}$$

$$\Delta F^{0\mathrm{D}} = c_0 \varepsilon_{\mathrm{cond}} dwL (1 - I/I_c)^{3/2}$$

Free energy barrier

$$\boxed{\Delta F^{2D} = c_2 \varepsilon_{\rm cond} d\xi^2 (1 - I/I_c)^{3/4}} \qquad \sim \frac{L_x}{L}\frac{L_y}{w}\Delta F^{0D} \qquad \frac{6}{4} - \frac{1}{4} - \frac{1}{2}$$

$$c_2 = 2^{27/4} 3^{-9/4} \pi = 28.55$$

$$\Delta F^{1D} = c_1 \varepsilon_{\rm cond} dw\xi (1 - I/I_c)^{5/4} \qquad \sim \frac{L_x}{L}\Delta F^{0D} \qquad \frac{6}{4} - \frac{1}{4}$$

$$\Delta F^{0D} = c_0 \varepsilon_{\rm cond} dwL (1 - I/I_c)^{3/2} \qquad \frac{6}{4}$$

Length scales:

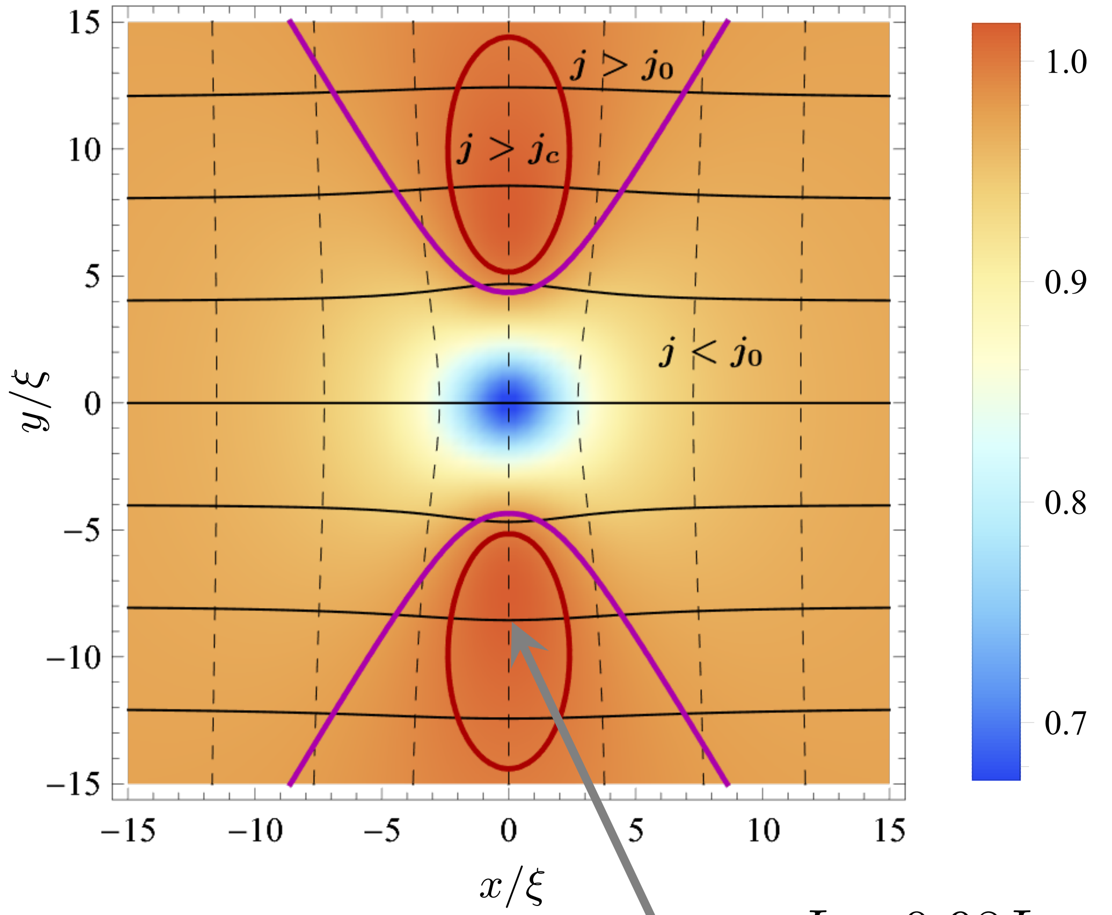
$$L_x \sim \frac{\xi}{\sqrt{\varepsilon}} \sim \frac{\xi}{(1 - I/I_c)^{1/4}} \qquad L_y \sim \frac{\xi}{\varepsilon} \sim \frac{\xi}{(1 - I/I_c)^{1/2}}$$

Supercurrent pattern at the saddle point

Supercurrent flow pattern at f_B:

$$\frac{j_x}{j_0} = 1 - \frac{18\varepsilon^2(2\varepsilon x^2 - 3\varepsilon^2 y^2 + 3)}{(2\varepsilon x^2 + 3\varepsilon^2 y^2 + 3)^2}$$

$$\frac{j_y}{j_0} = -\frac{72\varepsilon^3 xy}{(2\varepsilon x^2 + 3\varepsilon^2 y^2 + 3)^2}$$



Current at the instanton center:

$$\frac{j_B(0)}{j_0} = 1 - 6\varepsilon^2$$

formally vanishes at $I = 0.926 I_c$

$I = 0.98 I_c$

$\varepsilon = 1 - 3A^2$

$j = (1 - A^2)A$

narrow region of applicability

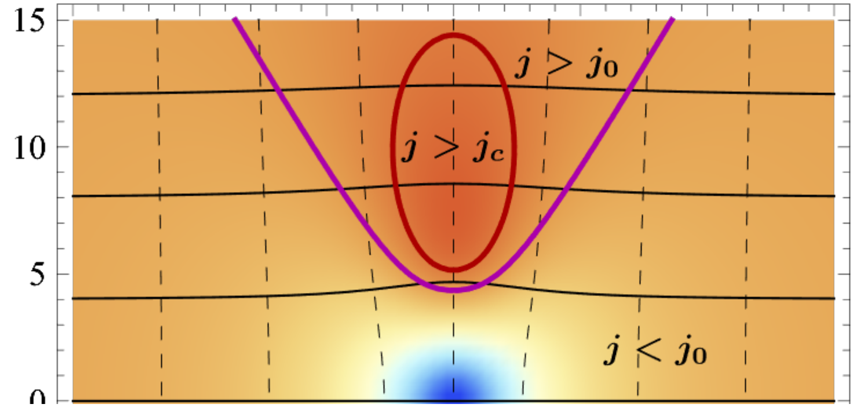
$I_{max} = 1.02 I_c$

Thermal phase slips in wide strips

- Wide strips $w \gg L_y$

  Instanton is formed near the edge

$$\Delta F = \frac{1}{2}\Delta F^{2\text{D}}$$



- Not so wide strips $w \ll L_y$

  1D LA instanton

$$\Delta F = \Delta F^{1\text{D}}$$

- Superconducting photon detectors
- Thermal phase slips
  - 0D: Josephson junction
  - 1D: Langer-Ambegaokar theory
  - 2D: Status quo
- Analytical solution in 2D at $I \to I_c$
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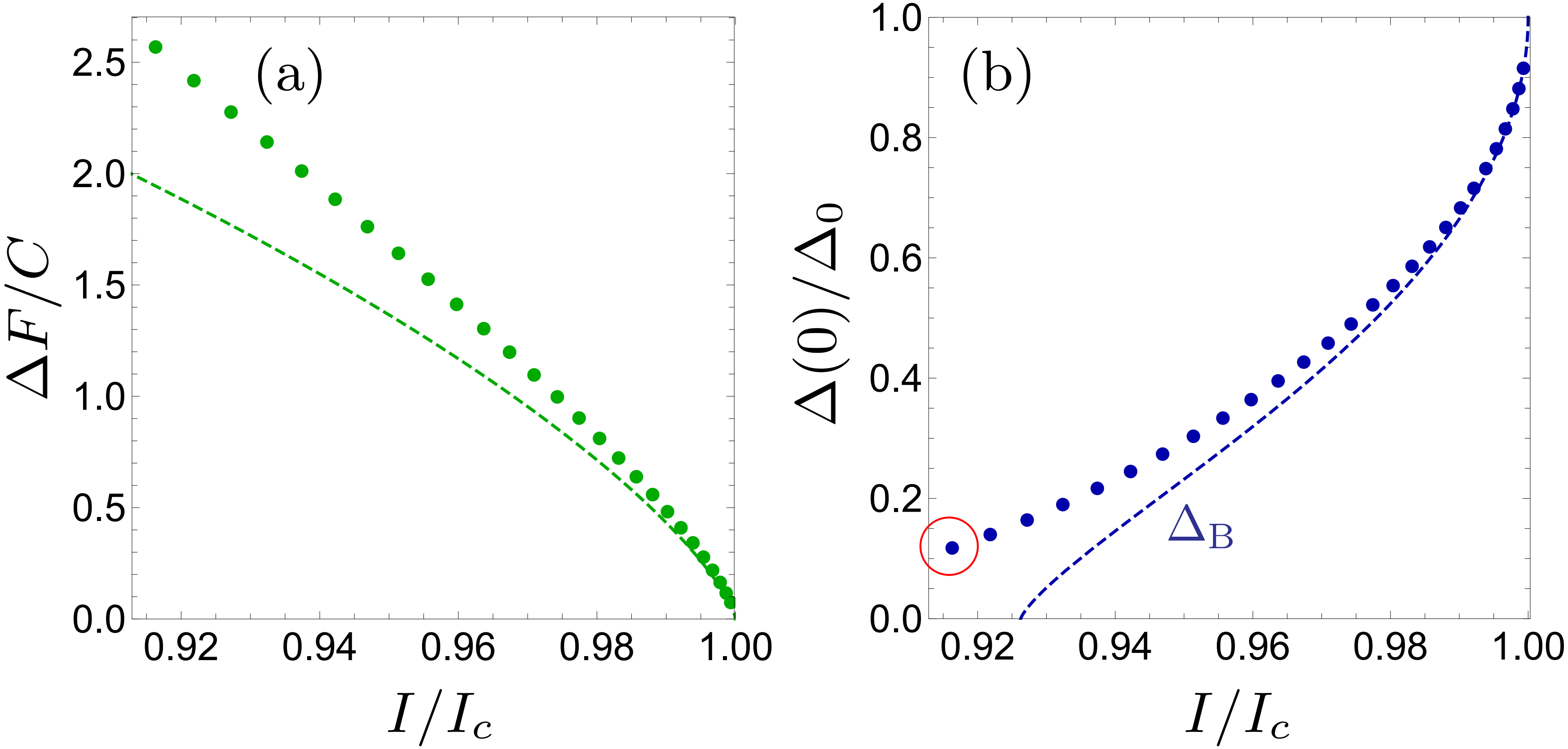
Variational approach at finite ε

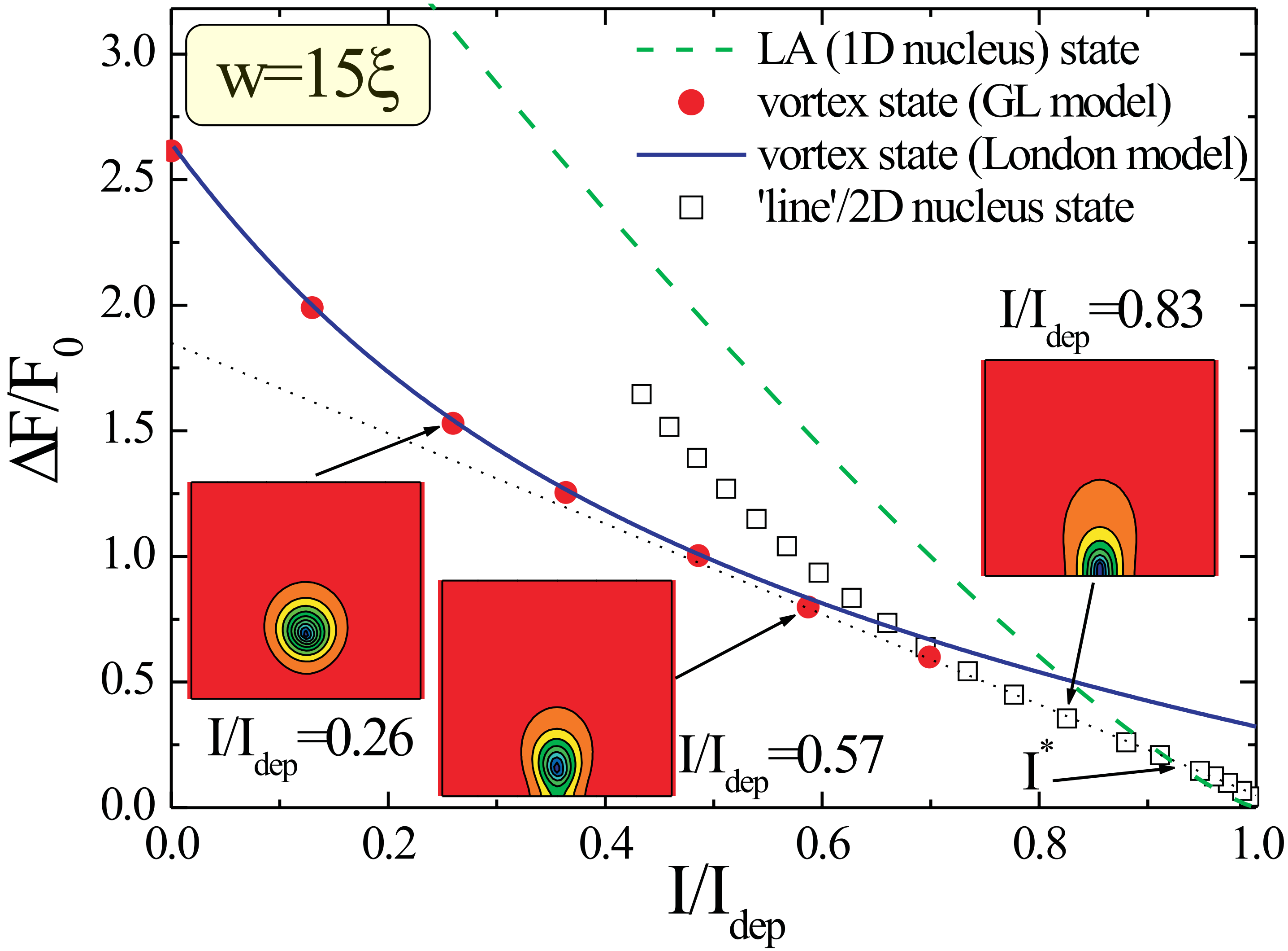
Boussinesq instanton: $f_{\rm B}(x,y) = -3\ln(1+2\varepsilon x^2/3+\varepsilon^2 y^2)$ $\qquad (I\to I_c)$

Boussinesq-like ansatz: $f(x,y) = -a\ln(1+c_1^2x^2+c_2^2y^2)$



GL instanton in 2D strips: numerics

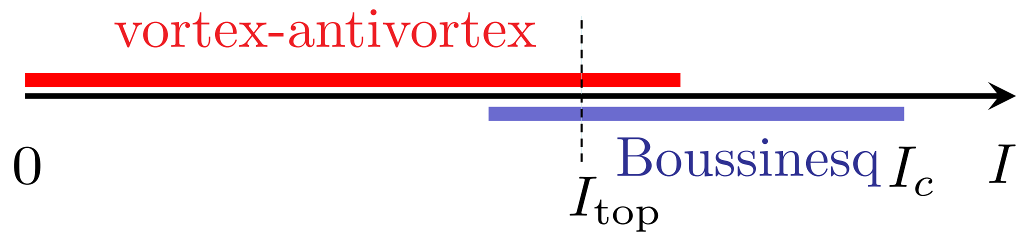
D. Y. Vodolazov (2012)



Topological transition between instantons

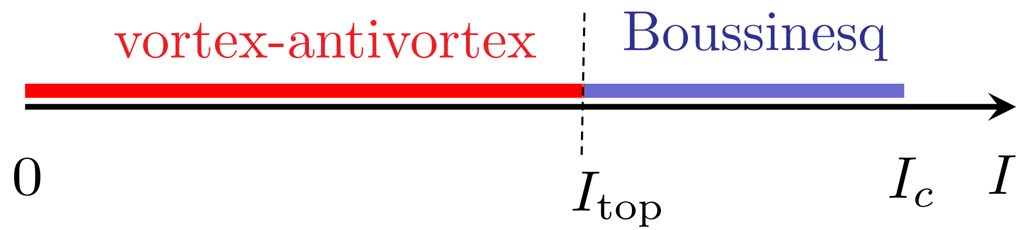
- **First order scenario**

  VA and B instantons coexist in some region of I
  Solution with the lowest ΔF wins

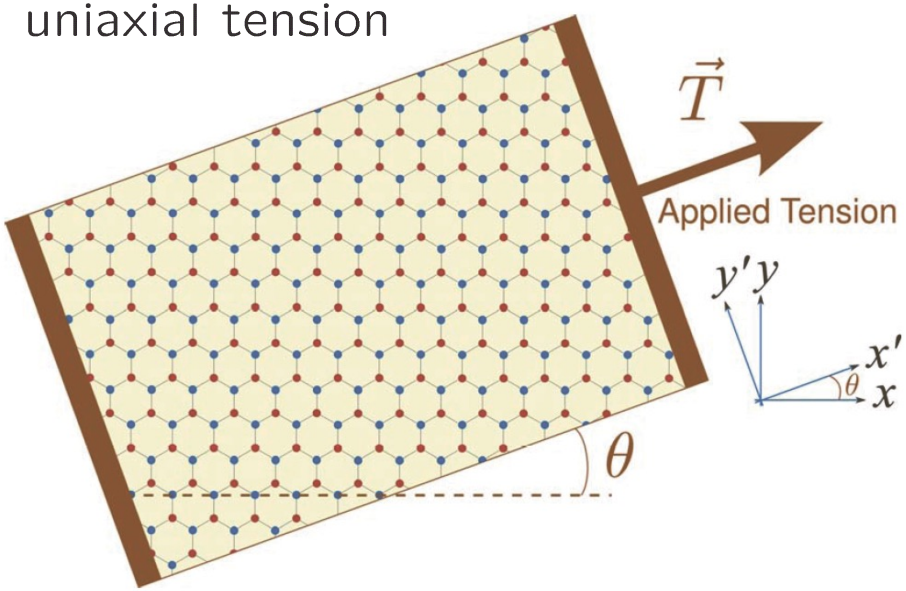


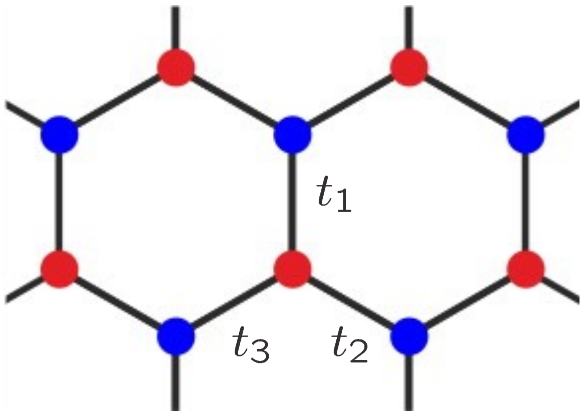
- **Second order scenario**

  VA and B instantons continuously transform to each other



Topological transition in stretched graphene

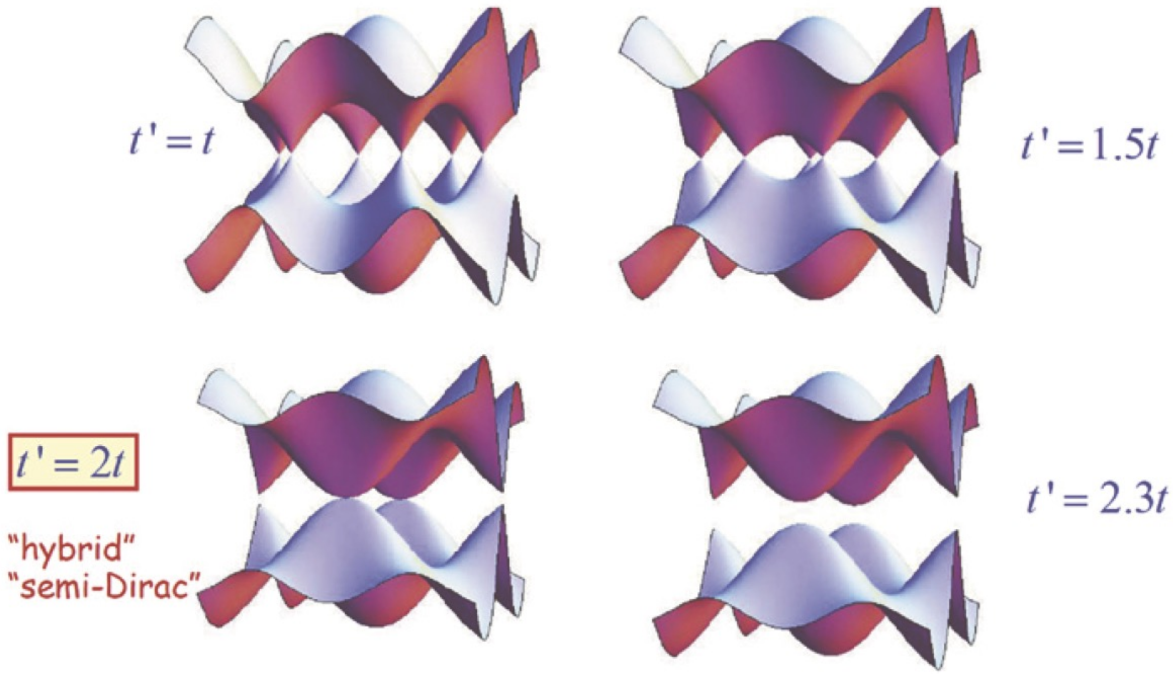




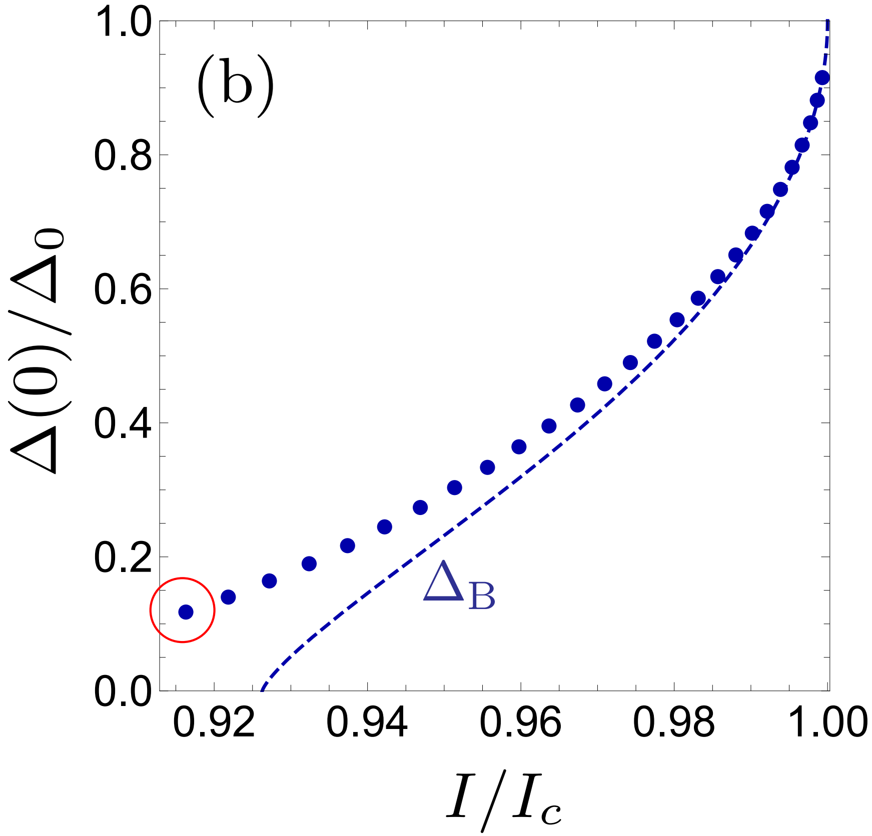
Toy model:
$t_1 = t',\ t_2 = t_3 = t$

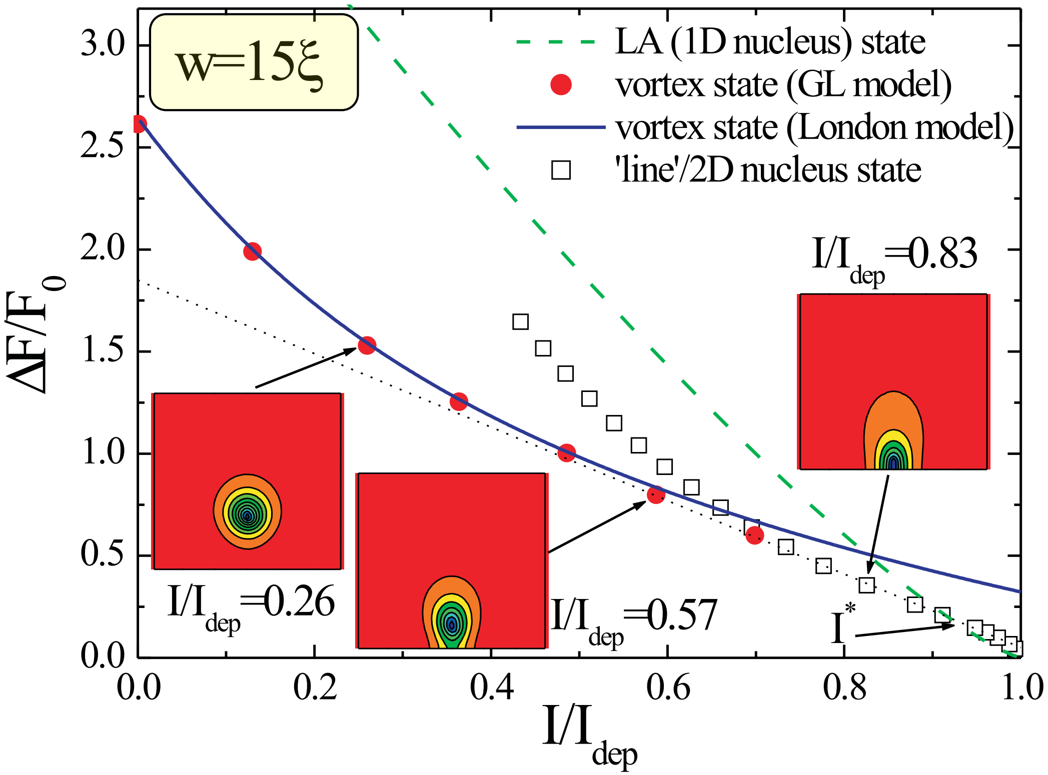
$$E(\mathbf{k}) = \pm\left|t' e^{i\mathbf{k}\boldsymbol{\delta}_1} + t e^{i\mathbf{k}\boldsymbol{\delta}_2} + t e^{i\mathbf{k}\boldsymbol{\delta}_3}\right|$$

G. Montambaux et al. (2009)
V. M. Pereira et al. (2009)

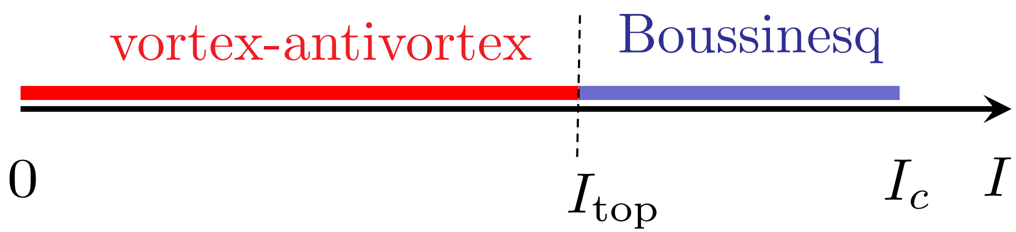


Topological transition between instantons





Second order transition appears most plausible



Summary

- First analytical solution for the GL instanton in 2D
- Obtained in the limit $I \to I_c$
    - Stream function
    - Elliptic Boussinesq equation
    - Hirota's form of the solution
- Second order topological transition expected at $I_{\rm top} \approx 0.9\, I_c$

MAS and A. V. Polkin, arXiv:2506.18130

Boussinesq instanton: variational approach

$$S = \int d\bar{x}\, d\bar{y} \left(\frac{1}{2} \bar{f}^2_{\bar{x}\bar{x}} + \frac{1}{2} \bar{f}^2_{\bar{x}\bar{x}\bar{x}} + \frac{1}{2} \bar{f}^2_{\bar{x}\bar{y}} + \frac{1}{3} \bar{f}^3_{\bar{x}\bar{x}} \right)$$

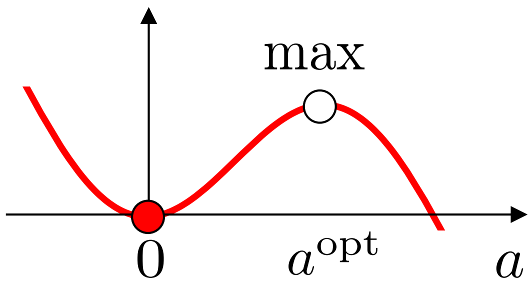
One can look for the saddle point of S using the ansatz $\bar{f} = az(\bar{x}, \bar{y})$, where the spatial profile $z(\bar{x}, \bar{y}) = \zeta(c_1\bar{x}, c_2\bar{y})$ determined by a probe function $\zeta(\cdot, \cdot)$ with $\zeta(0,0) = 1$. Optimization parameters: a and $c_{1,2}$.

- First we maximize with respect to the amplitude:

$$a^{\text{opt}} = -\frac{\langle z^2_{\bar{x}\bar{x}} + z^2_{\bar{x}\bar{x}\bar{x}} + z^2_{\bar{x}\bar{y}} \rangle}{\langle z^3_{\bar{x}\bar{x}} \rangle} > 0 \qquad \text{where } \langle \cdots \rangle = \int d\bar{x}\, d\bar{y}\, (\cdots)$$

- Then we minimize with respect to $c_{1,2}$:

$$S^{\text{opt}}[z] = \min_{c_1, c_2} \frac{\langle z^2_{\bar{x}\bar{x}} + z^2_{\bar{x}\bar{x}\bar{x}} + z^2_{\bar{x}\bar{y}} \rangle^3}{6 \langle z^3_{\bar{x}\bar{x}} \rangle^2}$$



Boussinesq instanton: variational approach

Ansatz for $\bar{f}(\bar{x},\bar{y})$	$c_1^{\rm opt}$	$c_2^{\rm opt}$	$S^{\rm opt}/8\pi$
$-6\ln(\bar{x}^2+\bar{y}^2+3)$			1
$a/(1+c_1^2\bar{x}^2+c_2^2\bar{y}^2)^{1/2}$	0.4	0.4	1.120
$a/(1+c_1^2\bar{x}^2+c_2^2\bar{y}^2)$	0.322	0.322	1.286
$a/(1+c_1^2\bar{x}^2)(1+c_2^2\bar{y}^2)$	0.298	0.422	1.612
$a/\cosh(c_1\bar{x})\cosh(c_2\bar{y})$	0.459	0.543	1.616
$a\exp(-c_1^2\bar{x}^2-c_2^2\bar{y}^2)$	0.365	0.365	3.003

- $S^{\rm opt} > 8\pi$ for all probe functions
- power-law functions with a small exponent provide a better approximation
- the best candidates depend on $\bar{x}^2+\bar{y}^2$

All these facts confirm $\bar{f}_{\rm B} = -6\ln(\bar{x}^2+\bar{y}^2+3)$ as the saddle-point solution with the minimal energy